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Some comments on the higher order theories of piezoelectric, piezothermoelastic and thermopiezoelectric rods and shells

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Abstract

In this note, the derivations of the higher order, 1-D (or 2-D), theories are discussed for the dynamic analysis of electroelastic (i.e., piezoelectric, piezothermoelastic and thermopiezoelectric) structural elements of uniform cross-section (or uniform thickness). Certain oversights are clarified concerning the higher order theories, including their variational formulation, invariant form and uniqueness of solutions that obscure the availability of earlier contributions in the open literature. In this respect, a higher order theory with some applications by Wu et al. most recently appeared in this journal [Int. J. Solids Struct. 39 (2002) 5325] is mentioned as one of the examples.

Keywords: Higher order theory; Electroelastic structural elements; Variational principles; Uniqueness of solutions

1. Introduction

Structural elements, rods and shells, are mathematical models for continua having their two or one dimensions much smaller than the third. In mathematically predicting the physical response of elements, some special types of one- and two-dimensional (1-D and 2-D) theories that may be traced back to the eighteenth century were proposed with little or no reference to their third dimension. The 1-D and 2-D theories that were mainly bound to linear elasticity were formulated under the well-known *ad hoc*, Bernoulli-Euler, Lagrange and Love and alike hypotheses of elements. The theories were referred, for instance, to Todhunter and Pearson (1886), Ericksen and Truesdell (1958) and Truesdell (1960). The resulting theories are naturally lacking in increasing accuracy and estimation of errors, but they are still in use due to their simplicity and clarity. Apparently, some other techniques were needed in deriving systematically 1-D and 2-D rational theories rather than by the *ad hoc* hypotheses. At the beginning of the nineteenth century, several French mathematicians sought to obtain the 2-D theories of elastic plates from the linear 3-D theory of elasticity. As noted by Habip (1965), von Kármán who presented his well-known theory of elastic plates without proof in 1910 indicated that a direct reduction of the 3-D theory to 1-D and 2-D theories is a very

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Nomenclature

x_i fixed, right-handed system of Cartesian (or Cartesian convected) coordinates
θ^i fixed, right-handed system of geodesic normal (and convected) coordinates
t time
$(\mathbf{g}_i, \mathbf{g}^i)$, $(\mathbf{a}_i, a^i)$ base vectors of space and the middle surface of shell
d, L characteristic length of cross-section, length of rod
h, $R_{\min}$ thickness of shell, least principal radius of curvature of the middle surface of shell
Θ, Θ_0 temperature increment, reference temperature
ε_r $(= d/L)$ rod parameter ($\ll 1$)
ε_s $(= h/|R_{\min}|)$ shell parameter ($\ll 1$)
ε_t $(= \Theta/\Theta_0)$ temperature parameter ($\ll 1$)
u_i, $\bar{u}_i$ components and shifted components of mechanical displacements
$u_i^{(m,n)}$, $u_i^{(n)}$ mechanical components of order (m, n) and (n)
ϕ, $\phi_{(m,n)}$, $\phi_{(n)}$ electric potential, electric potential of order (m, n) and (n)
$\Theta_{(m,n)}$, $\Theta_{(n)}$ temperature increment of order (m, n) and (n)
C_{mn} functions with derivatives of order up to and including (m, n) with respect to (x_i or θ^i, t)

important topic of future research, in his 1939 *Josiah Willard Gibbs* lecture given before the American Mathematical Society. Perhaps, motivated by von Kármán's lecture and influence, a large number of 1-D and 2-D theories were developed especially in third quarter of the twentieth century.

2. One- and two-dimensional theories

In predicting the physical response of a structural element, its 3-D theory is obviously more accurate but less tractable in analytical treatment than its 1-D (or 2-D) theory. In the 1-D (or 2-D) theory, the field variables are assumed to be not varied widely over the cross-section of rod (or across the thickness of shell), and hence, they may be averaged over the cross-section (or across the thickness), whereas in the 3-D theory, no restrictions are imposed. Both the 1-D (or 2-D) theory and the 3-D theory naturally contain some inevitable errors of experimental nature due to the constitutive modelling of element material. This type of errors cannot be measured or be reduced by simply increasing the accuracy of computation. Besides, some errors always remain in applications as a result of the rate and type of loading and the prescription of boundary and initial conditions as well as the method of computation employed. The relative merit of using either a 1-D (or 2-D) theory or a 3-D theory of an element basically depends on the element parameter, that is, the rod parameter (or the shell parameter), in each specific application. The rod parameter is defined by ε_r $(= d/L \ll 1)$ where d is a characteristic length of the cross-section and L is the length of the rod. The shell parameter is given by ε_s $(= h/|R_{\min}|)$ where h is the thickness of shell and $R_{\min}$ denotes the least principal radius of the middle surface of shell). However, the 1-D (or 2-D) theory is generally more desirable and thus employed in investigating the physical response in the presence of coupled mechanical, electrical and thermal effects. Actually, considerable effort continued to grow in deriving the 1-D and 2-D theories for structural elements made up of time-dependent and/or temperature-dependent and even moisture-dependent anisotropic materials. In spite of a large number of contributions, there remain still unanswered questions in the 1-D and 2-D theories, involving the existence of solutions (e.g., Dikmen, 1982; Ciarlet, 1998), the role of convergence (Oliveira, 1974), the estimation of errors (John, 1965) and, in particular, the effect of the rod (or shell) parameter that indicates the range of applicability in the 1-D (or 2-D) theory.

3. Method of reduction

A rod (or shell) theory for continua is extracted from the 3-D theory by various methods of reduction under the fundamental assumption of $\varepsilon_r \ll 1$ (or $\varepsilon_s \ll 1$). The fundamental assumption permits one to treat the rod (or shell) as a 1-D (or 2-D) mathematical model of the 3-D continua. Of the various methods of reduction [for a review of the methods, see Kil'chevskiy (1965), Gol'denveizer (1964, 1997), Naghdi (1972) and Pikul (2000)], the method of asymptotic integration and the method of series expansions due to Mindlin were widely employed in deriving the 1-D and 2-D theories of electroelastic elements. The method of asymptotic integration was proposed by Gol'denveizer and Lur'e (1947) in deriving the theories of elastic shells. Much later, it was applied to establish the theories of piezoelectric plates (e.g., Maugin and Attou, 1990), and rods and shells [e.g., Le (1999) who included a review of pertinent literature]. Mindlin's method of reduction was based on the series expansions of Cauchy (1828) and Poisson (1829), and the integral method of Kirchhoff (1850). By the method, the 3-D theory of a continuum may be systematically and consistently converted into a hierarchy of its 1-D (or 2-D) theory, including all the significant electroelastic effects as deemed necessary. Following the seminal paper of Tiersten and Mindlin (1962) on the higher order 2-D theory of piezoelectric plates, the method was successfully applied in establishing the higher order theories of piezoelectric rods, plates and shells. The higher order theories of electroelastic structural elements were reviewed, for instance, by Tiersten (1969), Dökmeci (1978, 1980, 1988a), Mikhailov and Parton (1990), Saravanos and Heyliger (1999), and Wang and Yang (2000).

4. Electroelastic 1-D and 2-D higher order theories

4.1. Assumptions

In deriving the 1-D (or 2-D) electroelastic theory of an element, that is, ε_r or $\varepsilon_s \ll 1$, all the field variables are assumed to exist, single-valued and continuous functions of the space coordinates and time, under the suitable regularity and smoothness assumptions for the region of element with no singularities of any type. Accordingly, a kinematic type of hypotheses was usually invoked as a starting point of reduction due to the fact that the differentiation operation is generally simpler than the integration operation, and besides the compatibility type of equations were not needed. The mechanical displacements, the electric potential and the temperature increment were almost always chosen as the basic field variables in the derivation of 1-D and 2-D theories. Any other field variables (e.g., stresses, strains, energy, electric displacements, electric fields, heat flux) may be chosen as a point of departure. The choice of basic field variables is a point of importance that is in need of further elaboration.

4.2. Rods

The basic field variables of a rod were selected as the mechanical displacements u_i, the electric potential ϕ and the temperature increment Θ of the form

$$[u_i(x_j,t), \phi(x_i,t), \Theta(x_i,t)] = \sum_{m,n=0}^{M,N} [u_i^{(m,n)}(x_3,t), \phi_{(m,n)}(x_3,t), \Theta_{(m,n)}(x_3,t)]x_1^m x_2^n \tag{1}$$

Here, the rod is referred to by a fixed, right-handed system of Cartesian [Cartesian convected for a thermopiezoelectric rod] coordinates x_i ($i = 1,2,3$) where the x_α-axes ($\alpha = 1,2$) are the principal axes of cross-section of, and the x_3-axis denotes the locus of centroids of cross-section of rod. Then, a system of 1-D higher order theory of piezoelectric rods (Dökmeci, 1974a; Chou et al., 1991, and for a review, see Mikhailov and Parton,

1990) and that of thermopiezoelectric rods (Askar Altay and Dökmeci, 2002a) were derived. The physical and geometrical non-linearity, the effect of second sound and the temperature dependency of a material are all incorporated into the higher order 1-D theory of thermopiezoelectric rods. The higher order theory is capable of predicting all the types of high- (or low-) frequency vibrations where the wavelengths are much smaller (or much larger) than the characteristic length of the cross-section of the rod.

4.3. Plates

The basic field variables of a plate were chosen, as in the case of rod, in the form

$$[u_i(x_j,t),\phi(x_i,t),\Theta(x_i,t)] = \sum_{n=0}^{N}[u_i^{(n)}(x_\alpha,t),\phi_{(n)}(x_\alpha,t),\Theta_{(n)}(x_\alpha,t)]P(x_3), \quad P(x_3) = x_3^n \tag{2}$$

where the plate is referred to by a fixed, right-handed system of Cartesian coordinates x_i where the x_α-axes are located on the middle plane of a plate and the x_3-axis is normal to the middle plane of plate. The function $P(x_3)$ was mostly used as power series (e.g., Tiersten, 1969; Mindlin, 1972, 1989), and also, as trigonometric series in a few studies (e.g., Bugdaycı and Bogy, 1981; Lee et al., 1987) and as Legendre polynomials (Batra and Vidoli, 2002). In Eq. (2), instead of the electric potential, Tiersten and Mindlin (1962) used the electric displacements. Recently, Wang and Yang (2000) published a comprehensive review article that summarised the development of higher order theories for piezoelectric plates and described their applications, as previously did Dökmeci (1980, 1988a) in his surveys and Tiersten (1969), Zelenka (1986) and Le (1999) in their treatises. Most recently, Tiersten (2002) dealt with the derivation of a higher order 2-D theory of electroted piezoelectric plates. As for thermopiezoelectric plates, Mindlin (1974, 1989) derived the theory of high-frequency motions of crystal plates accounting for coupling of mechanical, electrical and thermal fields. In a system of fixed, right-handed Cartesian convected coordinates, Askar Altay and Dökmeci (2001) formulated a system of 2-D plate theory for thermopiezoelectric materials subjected to strong electric fields and large deflections and included the effect of second sound. The theory accommodate all the types of vibrations at low-frequency where the wavelength is large as compared with the thickness of plate and also at high-frequency where the wavelength is of the order of magnitude or smaller than the thickness. Also, Dökmeci and Altay (2003) reported a higher order theory of porous piezoelectric plates in both invariant, differential and variational forms.

4.4. Shells

The first attempt to derive the higher order theory of a piezoelectric shell was due to Dökmeci (1974b), as remarked, for instance, by Mikhailov and Parton (1990). He deduced the 2-D theory in invariant form from the 3-D theory by choosing the basic field variables as the mechanical displacements and the electric potential of the form, namely

$$[\bar{u}_i(\theta^j,t),\phi(\theta^i,t),\Theta(\theta^1,t)] = \sum_{n=0}^{N}[u_i^{(n)}(\theta^\alpha,t),\phi_{(n)}(\theta^\alpha,t),\Theta_{(n)}(\theta^\alpha,t)]P(\theta^3), \quad P(\theta^3) = (\theta^3)^n \tag{3}$$

where the functions $u_i^{(n)} \in C_{12}$, $\phi_{(n)} \in C_{10}$ and $\Theta_{(n)} \in C_{10}$ are unknown *a priori* and independent functions of order (n) to be determined. The shell is referred to by a fixed, right-handed system of geodesic normal coordinates θ^i where the θ^α-curves are located on the middle surface of shell, the θ^3-axis is normal to the middle surface and $\theta^3 = 0$ defines the middle surface. The components u_i and shifted components $\bar{u}_i$ (see Naghdi, 1963; Librescu, 1975) of mechanical displacements, that is, the components referred to the base vectors $(\mathbf{g}_i,\mathbf{g}^i)$ of space and those $[(\mathbf{a}_\alpha,\mathbf{a}_3);(\mathbf{a}^\alpha,\mathbf{a}^3)]$ to the base vectors of middle surface, are associated with one to another, namely

$$\mathbf{u} = u_i\mathbf{g}^i = u^i\mathbf{g}_i = \bar{u}_\alpha\mathbf{a}^\alpha + \bar{u}_3\mathbf{a}^3 = \bar{u}^\alpha a_\alpha + \bar{u}^3\mathbf{a}_3, \quad u_\alpha = \mu^\beta_\alpha\bar{u}_\beta, \;\; u^\alpha = (\mu^{-1})^\alpha_\beta u^\beta, \;\; u^3 = u_3 = \bar{u}^3 = \bar{u}_3 \tag{4}$$

$$\mu^\alpha_\beta = \delta^\alpha_\beta - \theta^3 b^\alpha_\beta, \quad \mu^\alpha_\sigma(\mu^{-1})^\sigma_\beta = \delta^\alpha_\beta, \;\; H = \tfrac{1}{2}b^\alpha_\alpha, \;\; K = |b^\alpha_\alpha| \tag{5}$$

Here, Einstein's summation convention is implied for all repeated Latin indices $(1,2,3)$ and Greek indices $(1,2)$. The μ^α_β denotes the shifters, and b^α_β, H and K stand for the mixed components of second fundamental form, the mean and Gaussian curvatures of middle surface of shell, respectively. In the case of a plate, the curvature effect vanishes, $b_{\alpha\beta} = 0$ and hence the shifters are reduced to the Kronecker deltas, $\mu^\alpha_\beta = \delta^\alpha_\beta$ and $u_i = \bar{u}_i$. The formulation of piezoelectric shell theory is in tensor notation and accordingly, they can be readily expressed in any particular coordinate system most suitable for the geometric configuration at hand. Most recently, in a fixed, right-handed system of geodesic normal convected coordinates, another higher order theory (Askar Altay and Dökmeci, 2002b) was developed for non-linear vibrations of thermopiezoelectric shells subjected to small temperature change and strong electric fields, under large deflections and including the effect of second sound. The higher order theory governs all the types of non-linear vibrations at both low-frequency and high-frequency, and its fully variational form allows one to make simultaneous approximations upon all the field variables of thermopiezoelectric shells. Evidently, the unified theory is in agreement with and recovers, as special cases, some of earlier 1-D and 2-D higher order theories of piezoelectric, piezothermoelastic and thermopiezoelectric elements.

4.5. Piezothermoelastic elements—uniqueness of solutions

In piezothermoelasticity (Nowinski, 1978; Haojiang and Weiqiu, 2001), a special case of thermopiezoelectricity, the piezoelectric field is taken to be uncoupled from the thermal field as in uncoupled thermoelasticity (e.g., Boley and Weiner, 1960). Thus, the higher order 1-D and 2-D theories in piezothermoelasticity are readily obtainable from those mentioned above in thermopiezoelectricity. In so doing, the equation of heat conduction is excluded and the terms involving the temperature increment are retained only in the constitutive relations. On the other hand, in all the higher order 1-D and 2-D theories of electroelastic elements, the existence and uniqueness of solutions, that is, the internal consistency of solutions, are of special importance. The uniqueness of solutions were well established by means of the classical energy argument and the boundary and initial conditions sufficient for the uniqueness were enumerated for the higher order theories of piezoelectric rods (Dökmeci, 1974a), plates (Tiersten, 1969; Mindlin, 1989) and shells (Dökmeci, 1974b). As for the thermopiezoelectric elements, the uniqueness of solutions was investigated for thermopiezoelectric rods (Askar Altay and Dökmeci, 2002a), plates (Mindlin, 1974, 1989) and shells (Askar Altay and Dökmeci, 2002b). However, the existence of solutions has yet to be investigated in solutions of the 3-D theory as well as those of higher order 1-D and 2-D theories.

5. Electroelastic variational principles

Variational principles, integral and differential types (i.e., with and without an explicit functional), with their key features were contrived in piezoelectricity and thermopiezoelectricity. They were especially used in systematically deriving the higher order 1-D and 2-D theories. In piezoelectricity, Tiersten and Mindlin (1962), Tiersten (1967, 1969) and EerNisse (1967) primarily develop some two-field variational principles that operate on the mechanical displacements and the electric potential, by use of Hamilton's principle. Dökmeci (1973) reported some variational principles operating on six field variables, including the jump conditions across an internal surface of discontinuity as well as the initial conditions. Also, Dökmeci (1988b) deduced some variational principles from Hamilton's principle by modifying it through an involutory (Friedrichs' or Legendre's) transformation in nonlinear piezoelectricity and Dökmeci (1990) derived a variational principle for piezoelectric continua under a bias. As for thermopiezoelectricity, Mindlin (1974, 1989) obtained a three-field variational principle that yields the divergence equations (i.e., the stress equations of motion, the charge equation of electrostatics and the equation of heat conduction) and the associated boundary conditions, an extended version of this principle was reported by Dökmeci (1978, 1980). Askar Altay and Dökmeci (1996) derived some variational principles for thermopiezoelectric continua with a fixed internal surface of discontinuity with the aid of Toupin's (1956) principle of virtual work in elastic dielectrics and the thermal field vector. The thermal field vector e_i $(= -\Theta_{,i}$, that is, the gradient of temperature increment), a new concept, similar to the electric field vector E_i $(= -\phi_{,i}$, that is, the gradient of electric potential) is introduced to maintain the consistency of the gradient equations of thermopiezoelectric fields. The unified variational principles generate the divergence equations, the gradient equations (i.e., the strain-mechanical displacement, electric field-electric potential and thermal field-temperature increment relations), the constitutive relations and the associated natural boundary conditions as well as the jump condition across the surface of discontinuity. Most recently, a generalised version of this principle was given for thermopiezoelectric continua subjected to large deflections, strong electric fields and temperature increment and also, including the effect of second sound (Askar Altay and Dökmeci, 2002b). A comprehensive account of the differential and integral types of dynamic variational principles, including their features, formulations and applications, was reported, for instance, by Tiersten (1969), Dökmeci (1978, 1980, 1988a,b) and Mikhailov and Parton (1990).

6. An example: a higher order theory for piezoelectric shells

In a most recent paper, Wu et al. (2002) developed a higher order theory of piezoelectric shells with graded material properties in the thickness direction, that is, the constitutive behavior was assumed to be dependent on the thickness coordinate [cf., for instance, a theory of vibrations of coated, thermopiezoelectric laminae with homogeneous material or heterogeneous material in the thickness direction by Dökmeci (1978) and Askar Altay and Dökmeci (2002b)]. The developed theory was extracted from the 3-D fundamental equations of piezoelectricity by use of Hamilton's principle [cf., for instance Tiersten (1969) and Dökmeci (1988a,b, 1974b) where the principle was used in deriving a higher order dynamic theory of piezoelectric shells]. It was called a higher order theory since the assumed electric potential contains both linear and quadratic terms and the assumed mechanical displacements involve linear terms. The theory may be compared with that derived by Dökmeci (1974b) who presented a higher order theory of piezoelectric shells in invariant form that contains both the mechanical displacements and the electric potential of order (n) and includes a theorem of uniqueness as well. The basic field variables (i.e., the mechanical displacements and the electric potential) were assumed to be the functions of the aerial coordinates and the thickness coordinate, only, though a dynamic theory is developed. Nevertheless, this higher order theory was used to study properly the electromechanical characteristics of a simply supported circular cylindrical shell subjected to applied sinusoidal static loads. Some numerical results were reported in detail for both homogeneous and inhomogeneous shells.

7. Conclusions

In relation to electroelastic continua, certain remarks are briefly stated on the higher order 1-D and 2-D theories of structural elements available in the open literature. In addition, the transition from 3-D theory to 1-D and 2-D theories and its relative merit are discussed. It is concluded that the higher order theories may be still valuable in many instances of applications despite some powerful method of computation. Emphasis is placed upon the best choice of the basic field variables and the invariant and fully variational forms in the higher order theories and the role of the rod or shell parameter in the transition. The methods of reduction, the integral and differential types of variational principles and the internal consistency in solutions of the higher order theories are elaborated. The uniqueness was well established in solutions of the 3-D as well as 1-D and 2-D theories of piezoelectricity and thermopiezoelectricity. However, the existence of solutions has yet to be investigated in all the theories. Moreover, some crucial parameters that delineate the dynamic response of elements [see Steele et al. (1995) for a discussion of the parameters] and, in particular, the rod (or shell) parameter that is a basic indication of the accuracy in the 1-D and 2-D theories still remains to be incorporated into the higher order theories. Lastly, as an example of some oversights, a higher order theory most recently appeared in this journal is mentioned.

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